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Spectral Measure of Structural Robustness in Complex Networks

Jun Wu, Mauricio Barahona, Yue-Jin Tan, and Hong-Zhong Deng

***Abstract*—We introduce the concept of natural connectivity as a measure of structural robustness in complex networks. The natural connectivity characterizes the redundancy of alternative routes in a network by quantifying the weighted number of closed walks of all lengths. This definition leads to a simple mathematical formulation that links the natural connectivity to the spectrum of a network. The natural connectivity can be regarded as an average eigenvalue that changes strictly monotonically with the addition or deletion of edges. We calculate both analytically and numerically the natural connectivity of three typical networks: regular ring lattices, random graphs, and random scale-free networks. We also compare the proposed natural connectivity to other structural robustness measures within a scenario of edge elimination and demonstrate that the natural connectivity provides sensitive discrimination of structural robustness that agrees with our intuition.**

***Index Terms*—Complex networks, graph spectra, natural connectivity, structural robustness.**

Acronym

ER	Erdös–Rényi.
SF	Scale free.
BA	Barabási–Abert.

Notation

G	(V, E): simple undirected graph.
V	Set of vertices.
E	Set of edges.
N	Number of vertices.
M	Number of edges.
d_i	Degree of vertex v_i.
$d_{\min}$	Minimum degree.
$d_{\max}$	Maximum degree.
$A(G)$	Adjacency matrix of G.
n_k	Number of closed walks of length k.
S	Weighted sum of numbers of closed walks.
λ_j	jth largest eigenvalue of $A(G)$.
$\bar{\lambda}$	Natural connectivity.
$\rho(\lambda)$	Spectral density.
$M_\lambda(t)$	Moment-generating function of $\rho(\lambda)$.
$G+\epsilon$	Graph obtained by adding an edge ϵ to G.
$\hat{n}_k$	Number of closed walks of length k in $G+\epsilon$.
$\hat{n}'_k$	Number of closed walks of length k in $G+\epsilon$ with ϵ.
$\hat{n}''_k$	Number of closed walks of length k in $G+\epsilon$ without ϵ.
$R_{N,2K}$	$2K$-regular ring lattice with N vertices.
I_n	Modified generalized Bessel functions.
C_N	Cycle graph with N vertices.
$G_{N,p}$	ER random graph with N vertices and edge density p.
R	Radius of the bulk of the spectrum.
$\langle k\rangle$	Average degree.
$p(k)$	Degree distribution.
γ	Scale exponent.
w_i	Expected degree of vertex v_i.
$\tilde{d}$	Second-order average degree.
$\kappa_V(G)$	Vertex connectivity of G.
$\kappa_E(G)$	Edge connectivity of G.
$a(G)$	Algebraic connectivity of G.
f_c^R	Critical removal fraction of vertices under random failure.
κ	Criterion for the disintegration of networks.

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I. Introduction

We are surrounded by networks. Networks with complex topology describe a wide range of systems in nature and society. Examples include the Internet, metabolic networks, electric power grids, supply chains, urban road networks, and the world trade Web among many others. In the past few years, the discovery of small-world [1] and SF properties [2] has stimulated a great deal of interest in studying the underlying organizing principles of various complex networks. The study of complex networks has become an important area of multidisciplinary research involving physics, mathematics, biology, social sciences, informatics, and other theoretical and applied sciences.

The function and performance of complex networks rely on their structural robustness, i.e., the ability to endure threats and survive accidental events. For example, modern society is dependent on its critical infrastructure networks: communication, electrical power, rail, and fuel distribution networks. Failure of any of these critical infrastructure networks can bring the ordinary activities of work and recreation to a standstill. Other examples of structural robustness arise in biological and social systems, including questions such as the stability of social organizations in the face of famine, war, or even changes in social policy. The structural robustness is an important factor that influences the network reliability [3]–[6]. Due to their broad range of applications, the structural robustness has become a central topic in complex networks and receives growing attention [7]–[12].

A simple and effective measure is essential for the study of structural robustness. Early measures were related to the basic concept of graph connectivity [13]. Vertex (edge) connectivity, defined as the size of the smallest vertex (edge) cut, determines the structural robustness of a graph to the deletion of vertices (edges) in the sense of preservation of its global connectedness. However, the vertex or edge connectivity only partly reflects the ability of graphs to retain certain degree of connectedness under deletion. Other improved measures were

TABLE I
CHARACTERISTICS OF EXISTING MEASURES OF STRUCTURAL ROBUSTNESS

Measures	Computational complexity	Accuracy
connectivity	$o(N^2M^2)$	accurate
super connectivity	NP-hard	accurate
conditional connectivity	NP-hard	accurate
toughness	NP-hard	accurate
scattering number	NP-hard	accurate
tenacity	NP-hard	accurate
expansion parameter	NP-hard	accurate
isoperimetric number	NP-hard	accurate
algebraic connectivity	$o(N^3)$	accurate
critical removal fraction	uncertain	inaccurate

introduced and studied, including super connectivity [14], conditional connectivity [15], fault diameter [16], toughness [17], scattering number [18], tenacity [19], expansion parameter [20], and isoperimetric number [21]. In contrast to vertex (edge) connectivity, these new measures consider both the cost of damaging a network and the extent to which the network is damaged. Unfortunately, from an algorithmic point of view, the problem of calculating these measures for general graphs is nondeterministic polynomial-time hard problem. This implies that these measures are of no great practical use in the context of complex networks.

Another remarkable measure to unfold the structural robustness of a network is the second smallest eigenvalue of the Laplacian matrix, also known as the algebraic connectivity. Fiedler [22] showed that the magnitude of the algebraic connectivity reflects how well connected the overall graph is, i.e., the larger the algebraic connectivity is, the more difficult it is to cut a graph into independent components. For a survey of the vast literature on algebraic connectivity, see [23]. However, the algebraic connectivity is too coarse a measure to capture important features of structural robustness of complex networks (note, for instance, that the algebraic connectivity is equal to zero for a disconnected graph). We discuss it in detail in Section IV.

An alternative formulation of structural robustness within the context of complex networks emerged from the random graph theory [24] and was stimulated by the work of Albert *et al.* [25]. Instead of a strict *extremal* property, they proposed a *statistical* measure, i.e., the critical removal fraction of vertices (edges) for the disintegration of a network, to characterize the structural robustness of complex networks. The disintegration of network performance is measured in terms of network performance. The most common performance measurements include the diameter, the size of the largest component, the average path length, and efficiency [26]. As the fraction of removed vertices (or edges) increases, the performance of the network will eventually collapse at a critical fraction. Although we can obtain the critical removal fraction for some special networks analytically [27]–[31], generally, this measure can only be calculated through simulations, and then, the computational complexity is determined by the calculation of network performance.

In order to facilitate a comparison between various existing measures, we present the characteristics of existing measures of structural robustness in Table I.

An intuitive notion of structural robustness can be expressed in terms of the redundancy of routes between vertices. If we consider a source vertex and a termination vertex, there may be several routes between them. When one route fails, the two vertices can still communicate through other alternative routes. It is intuitive that the more alternative routes present, the more robust the connectedness between the two vertices. This observation leads us to consider the redundancy of alternative paths as a measure of the structural robustness of networks, since this ensures that the connection between vertices remains possible in spite of damage to the network. Although it would be ideal to define this redundancy as the number of alternative routes of different lengths for all pairs of vertices, this measure is very difficult to calculate. However, the number of closed walks in a network is a good index for the number of alternative routes. In this paper, we propose a new structural robustness measure of complex networks based on the number of closed walks in the graph, which is easy to compute.

II. DEFINITION OF NATURAL CONNECTIVITY

A complex network can be viewed as a simple undirected graph $G(V,E)$, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. Let $N = |V|$ and $M = |E|$ be the number of vertices and the number of edges, respectively. Let d_i be the degree of vertex v_i. Let $d_{\min}$ be the minimum degree and $d_{\max}$ be the maximum degree of G. Let $A(G) = (a_{ij})_{N\times N}$ be the adjacency matrix of G, where $a_{ij} = a_{ji} = 1$ if vertices v_i and v_j are adjacent and $a_{ij} = a_{ji} = 0$ if otherwise. A walk of length k in a graph G is an alternating sequence of vertices and edges $v_0 e_1 v_1 e_2 \cdots e_k v_k$, where $v_i \in V$ and $e_i = (v_{i-1}, v_i) \in E$. A walk is closed if $v_0 = v_k$.

Closed walks are directly related to the subgraphs of the graph. For instance, a closed walk of length $k = 2$ corresponds to an edge, and a closed walk of length $k = 3$ represents a triangle. Note that a closed walk can be trivial, i.e., containing repeated vertices, leading to the length of a closed walk being infinite. The number of closed walks is an important index for complex networks. Here, we propose to measure the redundancy of alternative routes as the scaled number of closed walks of all lengths. Considering that shorter closed walks have more influence on the redundancy of alternative routes than longer closed walks and to avoid the divergence of the number of closed walks of all lengths, we scale the contribution of closed walks of length k by the factorial of k. That is, we consider the weighted sum of numbers of closed walks $S = \sum_{k=0}^{\infty}(n_k/k!)$, where n_k is the number of closed walks of length k, which can be obtained from the powers of the adjacency matrix

$$n_k = \mathrm{trace}(A^k) = \sum_{j=1}^{N} \lambda_j^k \tag{1}$$

where λ_j is the jth largest eigenvalue of $A(G)$. Specifically, note that $n_2 = \sum_j d_j = 2M$.

Using (1), we obtain

$$S = \sum_{k=0}^{\infty}\sum_{j=1}^{N}\frac{\lambda_j^k}{k!} = \sum_{j=1}^{N}\sum_{k=0}^{\infty}\frac{\lambda_j^k}{k!} = \sum_{j=1}^{N} e^{\lambda_j}. \tag{2}$$

Hence, the proposed weighted sum of closed walks of all lengths can be derived from the graph spectrum. We remark that (2) corresponds to the Estrada index of the graph [32], [33], which has been used in several contexts in graph theory, including subgraph centrality [34], bipartivity [35], and expansibility [36], [37]. Noting that S will be a large number for large N, we scale S and denote it by $\bar{\lambda}$

$$\bar{\lambda} = \ln(S/N) = \ln\left[\frac{1}{N}\sum_{j=1}^{N} e^{\lambda_j}\right] \tag{3}$$

which corresponds to an "average eigenvalue" of the graph adjacency matrix. We propose to call it *natural connectivity or eigenvalue* of the graph.

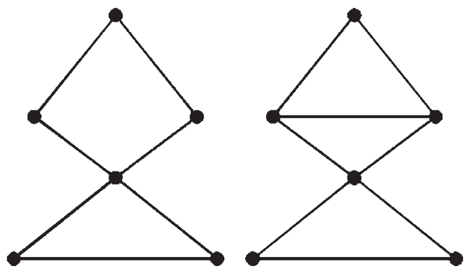

Fig. 1. Graph (b) is obtained from graph (a) by adding an edge. Both graphs have identical edge connectivity and identical algebraic connectivity but are distinguished by our proposed natural connectivity.

When $N \to \infty$, with continuous approximation for λ_i, (3) can be written as

$$\bar{\lambda} = \ln \left[\int_{-\infty}^{+\infty} \rho(\lambda) e^{\lambda} d\lambda \right] = \ln \left(M_{\lambda}(1) \right) \quad (4)$$

where $\rho(\lambda)$ is the spectral density and $M_{\lambda}(t)$ is the moment-generating function of $\rho(\lambda)$.

The proposed natural connectivity has some desirable features. In particular, $\bar{\lambda}$ changes monotonically when edges are added or deleted. To see this, consider a graph G where the number of closed walks of length k is n_k. Let $G + \epsilon$ be the graph obtained by adding an edge ϵ to G, and let $\hat{n}_k = \hat{n}'_k + \hat{n}''_k$ be the number of closed walks of length k in $G + \epsilon$, where $\hat{n}'_k (\hat{n}''_k)$ is the number of closed walks of length k with (without) ϵ. Note that $\hat{n}'_k \geq 0$ and $\hat{n}''_k = n_k$; hence, $\hat{n}_k \geq n_k$. It is easy to show that $\hat{n}_k > n_k$ for some k, e.g., $\hat{n}_2 = n_2 + 2$. Consequently, $\bar{\lambda}(G + \epsilon) > \bar{\lambda}(G)$, indicating that the natural connectivity increases strictly monotonically as edges are added.

Thus, the natural connectivity provides a sensitive means to detect the changes of structural robustness. For instance, consider the two simple graphs with six vertices in Fig. 1, where graph (b) is obtained by adding an edge to graph (a). Our intuition suggests that graph (b) should be more robust than graph (a). This agrees with our measure: The natural connectivities of graphs (a) and (b) are 1.0878 and 1.3508, respectively. However, some of the traditional structural robustness measures mentioned in the Introduction cannot distinguish the two graphs. For example, both graphs have identical edge connectivity, i.e., 2, and identical algebraic connectivity, i.e., 0.7369.

According to the monotonicity of natural connectivity, we know that, for a given number of vertices N, the empty graph (consisting of isolated vertices) has the minimum natural connectivity, and the complete graph (all of whose vertices are pairwise adjacent) has the maximum natural connectivity. Using the well-known results that $\lambda_1 = \lambda_2 = \cdots = \lambda_N = 0$ for the empty graph and $\lambda_1 = N - 1$, $\lambda_2 = \lambda_3 = \cdots = \lambda_N = -1$ for the complete graph [38], we obtain the following bounds for the natural connectivity:

$$0 \leq \bar{\lambda} \leq \ln \left[(N-1)e^{-1} + e^{N-1} \right] - \ln N \quad (5)$$

with asymptotic behavior as $N \to \infty$ given by

$$0 \leq \bar{\lambda} \leq N - \ln N. \quad (6)$$

III. Natural Connectivity of Typical Networks

The simplicity of the mathematical formulation of the natural connectivity introduced earlier enables us to obtain analytical results for different networks. We next derive the natural connectivity of three typical networks of relevance in different applications.

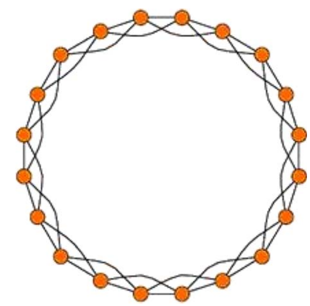

Fig. 2. Regular ring lattice with $N = 20$ and $K = 2$.

A. Natural Connectivity of Regular Ring Lattices

A regular ring lattice $R_{N,2K}$ is a $2K$-regular graph with N vertices on a 1-D ring in which each vertex is connected to its $2K$ neighbors (K on each side), as shown in Fig. 2. These graphs have been considered in the area of coordinated motion, proximity graphs, and synchronization [39]. The $R_{N,2K}$ constitutes the pristine worlds from which small-world graphs were obtained in the original Watts–Strogatz construction [1].

The adjacency matrix A of $R_{N,2K}$ is a symmetric circulant matrix of the form

$$A = \begin{bmatrix} c_0 & c_1 & \cdots & c_{N-1} \\ c_{N-1} & c_0 & \cdots & c_{N-2} \\ \cdots & \cdots & \cdots & \cdots \\ c_1 & c_2 & \cdots & c_0 \end{bmatrix} \quad (7)$$

where $c_k = 0$ if $k = 0$ or $K < k < N - K$ and $c_k = 1$ if $1 \leq k \leq K$ or $N - K \leq k \leq N - 1$. Diagonalization of this circulant matrix by the Fourier matrix leads to the following eigenvalues:

$$\lambda_j = \sum_{k=0}^{N-1} c_k \exp\left(-\frac{2\pi i k(j-1)}{N} \right), \qquad j = 1, 2, \ldots, N \quad (8)$$

where $i = \sqrt{-1}$.

For the regular ring lattices $R_{N,2K}$, the spectrum is then given by

$$\lambda_j = \sum_{k=1}^{K} 2\cos\left[\frac{2\pi k(j-1)}{N} \right], \qquad j = 1, 2, \ldots, N \quad (9)$$

which can be substituted into (3) to obtain the expression of the natural connectivity

$$\bar{\lambda}_{R_{N,2K}} = \ln \left[\frac{1}{N} \sum_{j=1}^{N} \exp\left(\sum_{k=1}^{K} 2\cos\left(\frac{2\pi k(j-1)}{N} \right) \right) \right]. \quad (10)$$

To simplify (10), we need to introduce the Bessel functions [40]. The Bessel functions of the first kind $J_{\alpha}(x)$ are defined as the solutions to the Bessel differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\alpha^2)y=0 \tag{11}$$

which are nonsingular at the origin. Another definition of the Bessel function for integer values of n is possible using an integral representation

$$J_n(x)=\frac{1}{\pi}\int_0^{\pi}\cos(n\tau-x\sin\tau)\,d\tau. \tag{12}$$

The generalized Bessel functions of the first kind for integer values of n are defined by

$$J_n(x_1,x_2,\ldots x_M)=\frac{1}{\pi}\int_0^{\pi}\cos(n\tau-x_1\sin\tau-x_2\sin 2\tau-\ldots-x_M\sin M\tau)d\tau. \tag{13}$$

A related class of functions is the modified Bessel functions of the first kind

$$I_\alpha(x)=i^{-\alpha}J_\alpha(ix) \tag{14}$$

which are the solutions to the modified Bessel differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(x^2+\alpha^2)y=0. \tag{15}$$

The corresponding modified generalized Bessel functions are defined by

$$I_n(x_1,x_2,\ldots x_M)=\frac{1}{\pi}\int_0^{\pi}\cos(n\tau)\exp\left[\sum_{s=1}^{M}x_s\cos s\tau\right]d\tau. \tag{16}$$

The following are some relevant properties of the modified generalized Bessel functions [41]–[43]:

$$I_n(x_1,x_2,\ldots x_M)=I_{-n}(x_1,x_2,\ldots x_M) \tag{17}$$

$$I_n(x_1,x_2,\ldots,x_M)\to 0,\qquad \text{as}\quad n\to\infty \tag{18}$$

$$\sum_{n=-\infty}^{\infty}e^{in\tau}I_n(x_1,x_2,\ldots,x_M)=\exp\left[\sum_{s=1}^{M}x_s\cos(s\tau)\right] \tag{19}$$

$$I_n(x_1,x_2,\ldots,x_M)=\sum_{\ell=-\infty}^{\infty}I_{n-M\ell}(x_1,x_2,\ldots,x_{M-1})I_\ell(x_M). \tag{20}$$

Using (19), we can rewrite (10) as

$$\begin{aligned}\bar{\lambda}_{R_{N,2K}}&=\ln\left[\frac{1}{N}\sum_{j=1}^{N}\sum_{n=-\infty}^{\infty}\exp\left(in\frac{2\pi(j-1)}{N}\right)I_n(\overbrace{2,2,\ldots 2}^{K})\right]\\&=\ln\left[\sum_{n=-\infty}^{\infty}I_n(\overbrace{2,2,\ldots 2}^{K})\cdot\delta_{n,Nz}\right]\\&=\ln\left[I_0(\overbrace{2,2,\ldots 2}^{K})+2\sum_{z=1}^{\infty}I_{Nz}(\overbrace{2,2,\ldots 2}^{K})\right]\end{aligned} \tag{21}$$

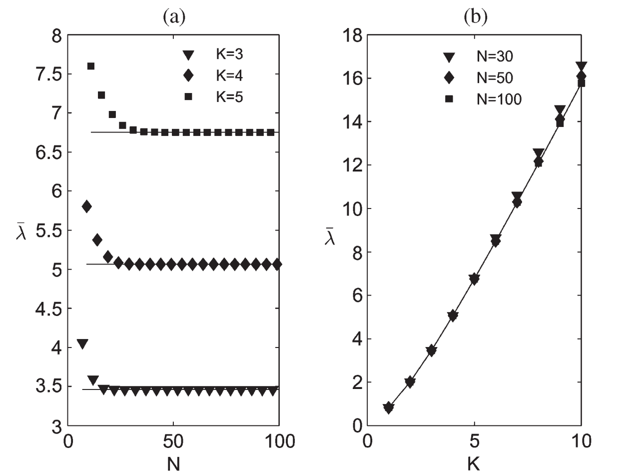

Fig. 3. Natural connectivity of regular ring lattices. (a) $\bar{\lambda}$ versus N with $K=$ (▼) 3, (♦) 4, and (■) 5; (b) $\bar{\lambda}$. (b) $\bar{\lambda}$ versus K with $N=$ (▼) 30, (♦) 50, and (■) 100. The symbols represent the numerical results, and the lines represent the analytical results according to (23).

where we have used (17) and the fact that

$$\frac{1}{N}\sum_{j=1}^{N}\exp\left[in\frac{2\pi(j-1)}{N}\right]=\delta_{n,Nz},\qquad z\in\mathbb{Z} \tag{22}$$

where $\delta_{n,Nz}$ is the Kronecker delta, i.e., $\delta_{n,Nz}=1$ if $n=Nz$ and $\delta_{n,Nz}=0$ if otherwise. From the limit in (18), $I_{Nz}(\overbrace{2,2,\ldots,2}^{K})\to 0$ as $N\to\infty$, and we obtain the following asymptotic result:

Theorem 1: The natural connectivity of a regular ring lattice $R_{N,2K}$ is

$$\bar{\lambda}_{R_{N,2K}}=\ln\left[I_0(\overbrace{2,2,\ldots 2}^{K})+o(1)\right] \tag{23}$$

where $o(1)\to 0$ as $N\to\infty$.

Remark 1: The cycle graph C_N is a special case of a regular ring lattice with $K=1$, i.e., $C_N=R_{N,2}$. Hence, $\bar{\lambda}_{C_N}=\ln(I_0(2)+o(1))$.

Using the recursion in (20), we calculate $I_0(\overbrace{2,2,\ldots,2}^{K})$ in terms of the standard (univariate) Bessel functions $I_n(x)$ and show in Fig. 3 that our analytical results agree well with numerical calculations. The figure confirms that, when N is large, the natural connectivity for regular ring lattices is independent of the network size. We also show that the natural connectivity increases monotonically (but sublinearly) with the number of neighbors $2K$, which is also the edge connectivity for these graphs, i.e., $\bar{\lambda}_{R_{N,2K}}<\lambda_1(R_{N,2K})=2K$, which follows directly from the definition of the natural connectivity shown in (3).

B. Natural Connectivity of Random Graphs

The theory of random graphs was introduced by Erdös and Rényi [44]. A detailed review of random graphs can be found in the classic book by Bollobás [24]. Here, we consider the classic ER random graph $G_{N,p}$ with N vertices, in which each of the $C_N^2=N(N-1)/2$ possible edges occurs independently with probability p. It is well known that the largest eigenvalue λ_1 of $G_{N,p}$ is almost surely $Np[1+o(1)]$ provided that $Np\geq\ln N$[45], [46]. Moreover, according to

Wigner's semicircle law [47], as $N \to \infty$, the spectral density of $G_{N,p}$ converges to the semicircular distribution

$$\rho_{SC}(\lambda) = \begin{cases} \frac{2}{\pi}\sqrt{1-\left(\frac{\lambda}{R}\right)^2}, & |\lambda| \le R \\ 0, & |\lambda| > R \end{cases} \tag{24}$$

where $R = 2\sqrt{Np(1-p)}$ is the radius of the "bulk" of the spectrum that contains all eigenvalues other than λ_1.

We now use these expressions to obtain the asymptotic behavior of the natural connectivity of ER random graphs with $\ln N/N \le p \le 1 - \ln N/N$ as follows:

$$\begin{aligned} \bar{\lambda}_{\mathrm{ER}} &= \ln\left[\int_{-R}^{+R} \rho_{\mathrm{SC}}(\lambda)e^{\lambda}d\lambda + e^{\lambda_1}/N\right] \\ &= \ln\left[M_{\lambda}^{\mathrm{SC}}(1) + e^{Np}/N\right] \end{aligned} \tag{25}$$

where

$$M_{\lambda}^{\mathrm{SC}}(1) = \frac{2}{\pi}\int_{-R}^{+R}\sqrt{1-\left(\frac{\lambda}{R}\right)^2}\, e^{\lambda}\, d\lambda. \tag{26}$$

Substituting $\lambda = R\cos(\theta)$ into (26), we obtain

$$M_{\lambda}(1) = \frac{2}{\pi}\int_{0}^{\pi} e^{R\cos(\theta)}\sin^2(\theta)\, d\theta. \tag{27}$$

Note that [40]

$$I_{\alpha}(x) = \frac{(x/2)^{\alpha}}{\pi^{1/2}\Gamma(\alpha+1/2)}\int_{0}^{\pi} e^{x\cos(\theta)}\sin^{2\alpha}(\theta)\, d\theta \tag{28}$$

where $I_{\alpha}(x)$ is the modified Bessel function and $\Gamma(x)$ is the Gamma function. Then, we obtain that

$$I_1(R) = \frac{R}{\pi}\int_{0}^{\pi} e^{R\cos(\theta)}\sin^2(\theta)\, d\theta. \tag{29}$$

Using (29), we can simplify (27) as

$$M_{\lambda}(1) = 2I_1(R)/R. \tag{30}$$

Substituting (30) into (25), we obtain

$$\begin{aligned} \bar{\lambda}_{\mathrm{ER}} &= \ln\left[2\frac{I_1(R)}{R} + \frac{e^{Np}}{N}\right] \\ &= Np - \ln(N) + \ln\left[1 + 2\frac{NI_1(R)}{e^{Np}R}\right] \\ &= Np - \ln(N) + \ln\left[1 + f(p)\right] \end{aligned} \tag{31}$$

where

$$f(p) = \frac{2NI_1(R)}{Re^{Np}}. \tag{32}$$

The fact that the spectrum of ER graphs has a large gap with the bulk of the spectrum concentrated in the semicircle would indicate that the asymptotic behavior of the natural connectivity is dominated by the largest eigenvalue. This can be shown in more details by characterizing the asymptotic behavior of $f(p)$ as follows. First, we give two simple lemmas.

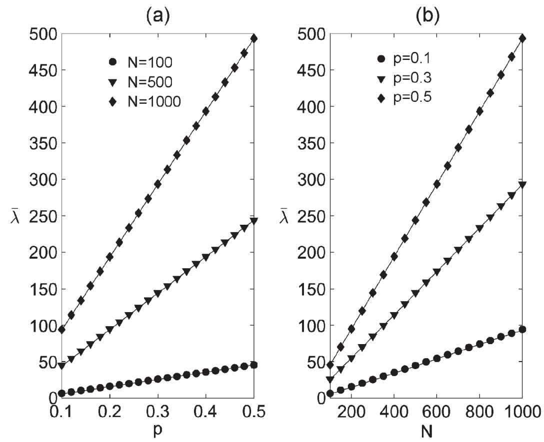

Fig. 4. Natural connectivity of ER random graphs. (a) $\bar{\lambda}$ versus p with $N =$ (●) 100, (▼) 500, and (♦) 1000. (b) $\bar{\lambda}$ versus N with $p =$ (●) 0.1, (▼) 0.3, and (♦) 0.5. Each quantity is an average over 1000 realizations. The lines represent the corresponding analytical results according to (39).

Lemma 1: As $N \to \infty$, $f(p)$ is a monotonically decreasing function for $\ln N/N < p < 1 - \ln N/N$.

Proof: It is easy to see that $2\sqrt{\ln N(1 - \ln N/N)} < R \le \sqrt{N}$ for $\ln N/N < p < 1 - \ln N/N$, which implies that $R \to \infty$ as $N \to \infty$. We can then use the asymptotic form of the modified Bessel functions $I_{\alpha}(x)$ valid for large $x \gg |\alpha^2 - 1/4|$[48]

$$I_{\alpha}(x) \to \frac{1}{\sqrt{2\pi x}}e^{x}. \tag{33}$$

Hence, for $\ln N/N < p < 1 - \ln N/N$

$$f(p) \to N\sqrt{\frac{2}{\pi}}\frac{e^{R-Np}}{R^{3/2}} \tag{34}$$

$$\frac{df(p)}{dp} \to N^2\sqrt{\frac{2}{\pi}}\frac{e^{R-Np}}{R^{5/2}}\left[(1-2p)\left(2-\frac{3}{R}\right) - R\right] \tag{35}$$

$$< 0. \tag{36}$$

Therefore, as $N \to \infty$, $f(p)$ is monotonically decreasing for $\ln N/N < p < 1 - \ln N/N$. ■

Lemma 2: Let $p_c = (\sqrt{\ln N + 1} + 1)^2/N$. Then, $f(p_c) \to 0$ as $N \to \infty$.

Proof: Note that $p_c \to \ln N/N$ from above and $p_c \to 0$ as $N \to \infty$. It then follows that

$$R(p_c) \to 2\left(\sqrt{\ln N + 1} + 1\right). \tag{37}$$

Hence

$$\begin{aligned} f(p_c) &\to N\sqrt{\frac{2}{\pi}}\cdot\frac{e^{R(p_c)-Np_c}}{R_{p_c}^{3/2}} \\ &= \frac{N}{2\sqrt{\pi}}\cdot\frac{e^{-\ln N}}{\left(\sqrt{\ln N+1}+1\right)^{3/2}} \\ &= \frac{1}{2\sqrt{\pi}\left(\sqrt{\ln N+1}+1\right)^{3/2}} \to 0. \end{aligned} \tag{38}$$

The proof is completed.

By Lemma 1 and Lemma 2, it is easy to see that $f(p) \le f(p_c) \to 0$ for $p_c \le p \le 1 - p_c$ as $N \to \infty$.

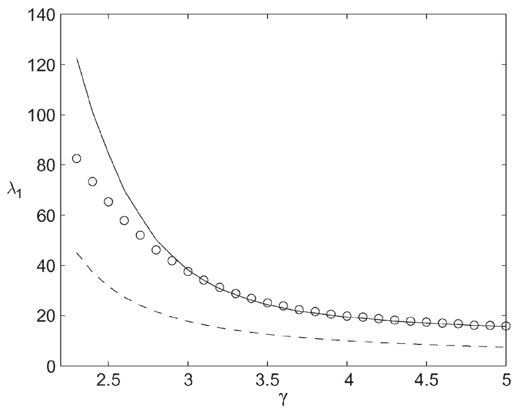

Fig. 5. (◯) Largest eigenvalue of random SF networks estimated by (solid line) the second-order average degree $\tilde{d}$ and (dashed line) the square root of maximum degree $\sqrt{M}$, where $N = 1000$ and $m = 10$. Each numerical result is an average over 100 realizations.

Consequently, we obtain the following result.

Theorem 2: Let $G_{N,p}$ be a random graph with $\zeta/N \leq p \leq 1 - \zeta/N$, where $\zeta = (\sqrt{\ln N + 1} + 1)^2$; then, the natural connectivity of $G_{N,p}$ almost surely is

$$\bar{\lambda} = Np - \ln N + o(1) \tag{39}$$

where $o(1) \to 0$ as $N \to \infty$.

Equation (39) shows that the natural connectivity of random graphs increases linearly with edge density p for a given graph size N. Since the average degree $\langle k \rangle \approx Np$, the natural connectivity of random graphs also increases linearly with the average degree. Fig. 4 shows that our analytical results agree well with the simulations of 1000 independent ER graphs for which we compute the average natural connectivity for each combination of N and p.

C. Natural Connectivity of Random SF Networks

SF networks play an important role in the field of complex networks. They have a power-law degree distribution $p(k) \sim k^{-\gamma}$ and have been found to describe many real-world networks in nature and society. Here, we study the random SF networks generated by the extended random graph model [49]. Note that, in contrast to the configuration model [50], the extended random graph model does not produce a graph with a prescribed degree sequence. Instead, it yields a random graph with a given expected degree sequence. We consider the random graphs with a given expected degree sequence $w_1 \geq w_2 \geq \cdots \geq w_N$, where $w_i = ci^{-1/(\gamma-1)}$, $\gamma > 2$. Here, c can be determined by the minimum degree $m = w_N = cN^{-1/(\gamma-1)}$; then, we obtain $c = mN^{1/(\gamma-1)}$. It also follows that the maximum degree $M = w_1 = mN^{1/(\gamma-1)}$. The vertex v_i is assigned with a vertex weight w_i. The edges are chosen independently and randomly with probability $\rho_{ij} = w_i w_j / \sum_t w_t$, and the expected degree of v_i is w_i. Notice that we allow loops in the model (for computational convenience) but their presence does not play any essential role. It is easy to verify that the degree distribution is $p(k) = (\gamma - 1)m^{\gamma-1}k^{-\gamma}$ and the average degree is $\langle k \rangle = m(\gamma - 1)/(\gamma - 2)$ [51].

Although there has been an extensive work on the spectral density of SF networks, it is still difficult to obtain all the eigenvalues analytically. From (3), we observe that the largest eigenvalue plays an important role for natural connectivity when the spectral gap is large. Our numerical observations indicate that this is the case for the SF networks studied here. This leads us to consider the following approximation for the natural connectivity:

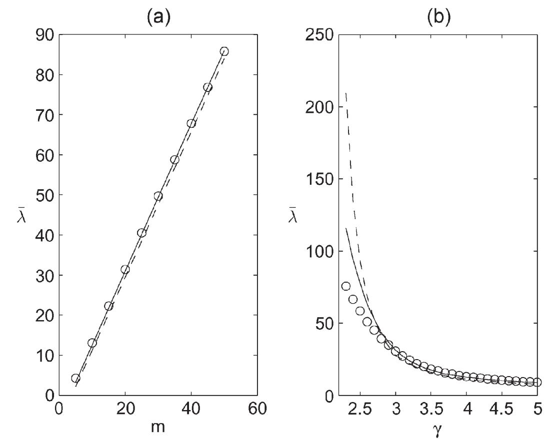

Fig. 6. (a) Natural connectivity of random SF networks as a function of minimum degree m, where $N = 1000$ and $\gamma = 4$. (b) Natural connectivity of random SF networks as a function of scale exponent γ, where $N = 1000$ and $m = 10$. The dashed line represents the estimation by (40), and the solid line represents the estimation by (41). (◯) Each numerical result is an average over 100 realizations.

$$\bar{\lambda}_{\mathrm{SF}} = \ln \left[\frac{1}{N} \left(\sum_{i=2}^{N} e^{\lambda_i} + e^{\lambda_1} \right) \right] \approx \lambda_1 - \ln N. \tag{40}$$

Chung *et al.* [45] proved that λ_1 is roughly equal to the second-order average degree $\tilde{d} = \langle k^2 \rangle / \langle k \rangle$ or the maximum degree $\sqrt{M}$*if* one of them is much larger than the other, where the notation $\langle . \rangle$ denotes the expectation. However, in most cases, $\tilde{d}$ and $\sqrt{M}$ are comparable. Fig. 5 shows the largest eigenvalue λ_1 along with $\tilde{d}$ and $\sqrt{M}$ as a function of the scale exponent γ. We find that λ_1 can be well estimated by $\tilde{d}$ if $\gamma \geq 3$. These results appear consistently for different N's and m's and will be discussed in a future publication.

To simplify our analytical calculations, we now consider random SF networks with $\gamma \geq 3$ and approximate the natural connectivity as

$$\bar{\lambda}_{\mathrm{SF}} \approx \lambda_1 - \ln N \approx \tilde{d} - \ln N \tag{41}$$

where

$$\tilde{d} = \begin{cases} m\frac{\gamma-2}{\gamma-3} = \langle k \rangle \frac{(\gamma-2)^2}{(\gamma-1)(\gamma-3)}, & \gamma > 3 \\ m\frac{\ln N}{2} = \langle k \rangle \frac{\ln N}{4}, & \gamma = 3. \end{cases} \tag{42}$$

In Fig. 6, we show both the numerical and estimated results from (40) and (41). We find that the natural connectivity can be estimated by (40) very well. Moreover, when $\gamma \geq 3$, it can be estimated by (41) with small errors, which come from the finite-size effect of networks. Equation (41) and our numerics show that the natural connectivity of random SF networks with $\gamma \geq 3$ increases (linearly) with m (or $\langle k \rangle$) for given N and γ and it decreases with γ for given N and m.

IV. Comparison With Other Measures of Robustness Under Edge Deletion

We now explore in depth the behavior of the natural connectivity under different scenarios of edge elimination and compare it with other existing structural robustness measures. Clearly, as edges are deleted, we expect a decrease of the structural robustness of the network. However, we also expect that different elimination strategies will lead to different behaviors in the collapse of the network. To test this, we

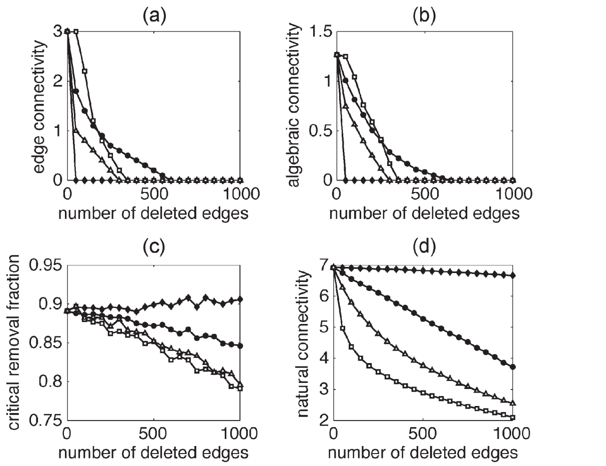

Fig. 7. Structural robustness measured by (a) edge connectivity, (b) algebraic connectivity, (c) critical removal fraction of vertices, and (d) natural connectivity, as a function of the number of deleted edges for four edge elimination strategies: (●) random strategy, (□) rich–rich strategy, (♦) poor–poor strategy, and (△) rich–poor strategy. The initial network is generated using the BA model, with $N = 1000$ and $\langle k \rangle \approx 6$. Each quantity is an average over 100 realizations. The lines are guide to the eye.

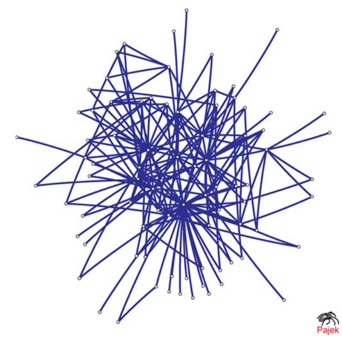

Fig. 8. Chinese Internet AS-level topology CN05, which contains 84 vertices and 211 edges.

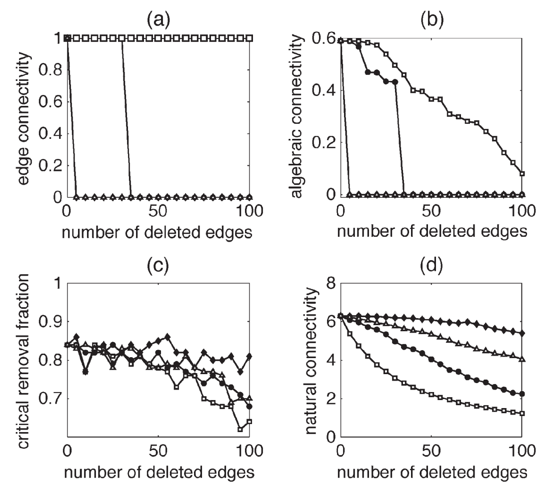

Fig. 9. Structural robustness measured by (a) edge connectivity, (b) algebraic connectivity, (c) critical removal fraction of vertices, and (d) natural connectivity, as a function of the number of deleted edges for four edge elimination strategies: (●) random strategy, (□) rich–rich strategy, (♦) poor–poor strategy, and (△) rich–poor strategy. The initial network is the Chinese Internet AS-level topology CN05. The lines are guide to the eye.

generate initial networks with a power-law degree distribution using the BA model [2], corresponding to SF networks with $\gamma = 3$, and consider four edge elimination strategies: 1) deleting the edges randomly (random strategy); 2) deleting the edges connecting high-degree to high-degree vertices in descending order of $d_i \cdot d_j$, where d_i and d_j are the degrees of the end vertices of an edge (rich–rich strategy); 3) deleting the edges connecting low-degree to low-degree vertices in ascending order of $d_i \cdot d_j$ (poor–poor strategy); and 4) deleting the edges connecting high- to low-degree vertices in descending order of $|d_i - d_j|$ (rich–poor strategy). We remark that the type of network chosen has no effect on the analysis and conclusions that follow.

Along with the natural connectivity, we investigate three other structural robustness measures: edge connectivity $\kappa_E(G)$, algebraic connectivity $a(G)$, and critical removal fraction of vertices under random failure f_c^R. To find the critical removal fraction of vertices, we use $\kappa \equiv \langle k^2 \rangle / \langle k \rangle \leq 2$ as the criterion for the disintegration of networks [27]. The results shown in Fig. 7 correspond to averages over 100 realizations of a BA network.

Our numerics in Fig. 7(a) and (b) show similar behavior for $\kappa_E(G)$ and $a(G)$. The first observation is that deleting a small quantity of rich-to-rich edges has no obvious effect on the structural robustness measured by the edge or algebraic connectivity. On the other hand, the structural robustness drops rapidly under the poor–poor strategy. It is generally believed that the edges between high-degree vertices are important, and the edges between low-degree vertices are inessential for the global network robustness. For example, in the Internet, the failure of the links between core routers will bring a disaster, but there is no effect on the structural robustness if we disconnect two terminal computers. Clearly, structural robustness measures based on edge or algebraic connectivity do not agree with our intuition. These unexpected features can be explained by the bound $a(G) \leq \kappa_V(G) \leq \kappa_E(G) \leq d_{\min}$, also known as Fiedler's inequality [22], where $\kappa(G)$ is the vertex connectivity. In fact, we find that the probability of $\kappa_E(G) = d_{\min}$ almost approaches one. The edge connectivity drops quickly in the poor–poor strategy since, after a few poor–poor edges are deleted, $d_{\min}$ decreases rapidly. On the other hand, $d_{\min}$ is preserved under the rich–rich strategy. Moreover, we find that, for all four strategies, the edge or algebraic connectivity is equal to zero after particular edges are deleted, even in the case where only very few vertices are separated from the largest cluster. This means that both the edge and algebraic connectivities lose discrimination when the network is disconnected.

Fig. 7(c) shows the critical removal fraction of vertices f_c^R as a function of the number of deleted edges. Contrary to the result of edge or algebraic connectivity and in agreement with our intuition, we observe that the rich–rich strategy is the most effective edge elimination strategy and the poor–poor strategy is the least effective to induce the collapse of the network. However, our numerics highlight the irregular behavior of the curves as edges are deleted even after averaging over many realizations. This indicates that the critical removal fraction is not a sensitive measure of structural robustness, particularly for small-sized networks.

In Fig. 7(d), we show the results of the natural connectivity according to (3). We find a clear variation of the measure with distinct differences between the four edge elimination strategies, showing a clear ranking for the four edge elimination strategies: rich–rich strategy $\succ$ rich–poor strategy $\succ$ random strategy $\succ$ poor–poor strategy, which agrees with our intuition. For the random strategy, we observe a linear decrease of the natural connectivity. For the rich–rich or rich–poor strategy, the natural connectivity decreases rapidly with the edge elimination. For poor–poor strategy, deleting a small quantity of poor–poor edges has a weak effect on the structural robustness. Moreover, the curves of the natural connectivity are smooth, a consequence of the strict monotonicity of the measure. This indicates that the natural connectivity can measure the structural robustness of complex networks stably even for very small sized networks. In fact, we find that the curves for natural connectivity are also smooth even without averaging over 100 realizations, viz., for each individual network. In contrast, in the case of individual networks, we find stepped curves for the edge or algebraic connectivity and large fluctuations for the critical removal fraction.

We repeat the aforementioned comparisons using the real data of Chinese Internet Autonomous System (AS)-level topology CN05 [52] shown in Fig. 8, which contains 84 vertices and 211 edges. We obtain similar results (see Fig. 9).

V. Conclusion

We have proposed the concept of natural connectivity as a spectral measure of structural robustness in complex networks. The natural connectivity is rooted in the inherent structural properties of a network and is expressed in mathematical form as an average eigenvalue. The theoretical motivation of our measure arises from the fact that the structural robustness of a network comes from the redundancy of alternative routes. The natural connectivity allows a precise quantitative analysis of the structural robustness and works both in connected and disconnected networks. We have proved that it changes strictly monotonically with the addition or deletion of edges. We have given the analytical expression of natural connectivity for three well-known networks: regular ring lattices, ER random graphs, and random SF networks. We have shown that the natural connectivity has strong analytical ability for these typical networks. To test our natural connectivity measure and compare it with other measures, we have designed a scenario of edge elimination, in which four different edge elimination strategies are considered. We have demonstrated that the natural connectivity has an acute discrimination in measuring the structural robustness of complex networks and can detect small variations of robustness stably.

Rich information about the topology and dynamical processes can be extracted from the spectral analysis of the networks. The natural connectivity sets up a bridge between graph spectra and the structural robustness of complex networks. The link between structural robustness and spectral graph theory is of great theoretical and practical significance in network design and optimization as it opens possibilities to connect structural robustness to other network structural or dynamical properties such as efficiency, synchronization, diffusion, and searchability. It is worthy to indicate that we just propose the concept of natural connectivity for simple undirected networks. The extension of natural connectivity for directed or weighted networks is still an open question.

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